About Causal Models and Quantum Stochastic Processes: Does Time-Delayed Feedback turn Markovian into non-Markovian Dynamics?

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The answer is: No! No feedback control loop turns a Markovian process into a non-Markovian one. The goal of this lecture will be to understand why that is the case. To do so, we need to talk about classical stochastic processes, and in particular the meaning of the Kolmogorov consistency condition. We will see that it is necessary to overcome this condition to understand feedback control. This leads to the much richer framework of causal models, which are able to distinguish between correlation and causation (which a classical stochastic process cannot). Interestingly, an understanding of classical causal models brings us very close to an understanding of a quantum stochastic process, describing the most general quantum dynamics respecting causality. The second part of the lecture is then devoted to providing an explicit theoretical framework for quantum stochastic processes and to introduce a meaningful operational Markov condition, which shows that feedback control does not change the Markovianity of the process.

CONTENTS

Main literature suggestions.

- Almost everything I am going to say is covered in Chapter 1 and Appendix A and B of Ref. [1], which can be downloaded for free from here.
- Another classic reference on classical causal models is the book by Judea Pearl [2].
- A recent tutorial on (mostly) quantum stochastic processes and causal models was written by Milz and Modi [3].

I. INTRODUCTION

It is a widespread habit to call 'time-local' differential equations of the form

$$\dot{x}(t) = R(t)x(t) \tag{1}$$

Markovian, whereas differential equations with a 'memory kernel',

$$\dot{x}(t) = \int_0^t ds K(t-s)x(s), \tag{2}$$

are called *non-Markovian*. That things are not that simple is easily recognized by assuming that the Green's function or propagator G of the dynamics, defined as $x(t) = G(t)x(0)$, is invertible. In that case we can immediately convert Eq. (2) into Eq. (1) by writing

$$\dot{x}(t) = \int_0^t ds K(t-s)G(s)x(0) = \int_0^t ds K(t-s)G(s)G(t)^{-1}x(t) \equiv R(t)x(t), \tag{3}$$

so is the dynamics Markovian or non-Markovian now?

Within this collaborative research center SFB 910 much attention is devoted to understanding *time-delayed feedback*, which is used to turn some dynamical system with equation $\dot{x}(t) = Rx(t)$ into a dynamical system of the form $\dot{x}(t) = Rx(t) + Fx(t-\tau)$ (F = feedback force, τ = delay parameter). Again, it is tempting to call the latter dynamics non-Markovian. However, the main message of this lecture will be to explain why *no feedback control law can change Markovian dynamics into non-Markovian dynamics*. Thus, if the original system $\dot{x}(t) = Rx(t)$ is Markovian, so is the situation described by $\dot{x}(t) = Rx(t) + Fx(t-\tau)$.

Why should this be true? Intuitively, we like to picture a dynamics as Markovian if the environment to which the system $x(t)$ is coupled is *memoryless*. Thus, (non-)Markovianity is a property of the *environment!* But any feedback control law acts on the *system* and thus, it cannot change

the nature of the environment. A memoryless environment remains memoryless independent of what 'we' (or the external agent) decide to do with the system.[1]

II. CLASSICAL STOCHASTIC PROCESSES

To make our intuition above rigorous and to provide a general definition of Markovianity, we first review some basics about stochastic processes, which we want to overcome later on. We do not attempt any high-level mathematical definition of a stochastic process here. Instead, we simply imagine some system of interest that we observe at an arbitrary set of discrete times $t_n > \cdots > t_1 > t_0$. This system is described by some set of states $x \in X$, which we assume to be finite for simplicity. Examples: "heads" and "tails" of a coin, the discretized position of a Brownian particle, the number of photons in a cavity, the fraction of people paying attention to this lecture, etc. If x_k denotes the state of the system at time t_k, then

$$p(x_n, t_n; \ldots; x_1, t_1; x_0, t_0) \tag{4}$$

denotes the probability to find the system in state x_0 at time t_0, then in state x_1 at time t_1, and so on and so forth until we find it in state x_n at time t_n. Importantly, a probabilistic description is in general necessary because the state x_k at time t_k does not uniquely determine the states at earlier or later times owing to the presence of an external and uncontrollable "environment". Examples: the surrounding water molecules in case of the Brownian particle, the (unknown) distribution of how much sleep the audience got in the night prior to this lecture, etc.

Notation. Since the index on x is in one-to-one correspondence with the index on t, I will also write Eq. (4) as $p(x_n, \ldots, x_1, x_0)$. In fact, to save even further space I introduce the sequence $\mathbf{x}_n = (x_n, \ldots, x_1, x_0)$ of measurement results and also write $p(\mathbf{x}_n)$.

Importantly, the $(n+1)$-time joint probability distribution (4) can be reconstructed experimentally from a high-dimensional histogram by repeating the experiment many (in fact, *very many* for

[1] One might argue that there are situations where one also applies feedback control to parts of the environment, thereby potentially changing its properties. The answer to that objection is that one then better labels this part of the environment as part of the *system* too. That is, we adapt *the convention that the environment is by definition the part of the world that we cannot control.*

large n) times. Obviously, to be a valid probability distribution, Eq. (4) has to satisfy

$$p(\mathbf{x}_n) \geq 0 \quad \forall \mathbf{x}_n \quad \text{and} \quad \sum_{\mathbf{x}_n} p(\mathbf{x}_n) = 1. \tag{5}$$

However, there is another important property that Eq. (4) has to satify known as the **Kolmogorov consistency condition (KCC)**. To explain it, imagine we repeat the same experiment again and measure the system at the same times *except* at time t_k with $k \in \{0, 1, \ldots, n\}$. Let us denote the resulting n-time joint probability distribution by $p(x_n, \ldots, \cancel{x_k}, \ldots, x_0)$. Then, we require

$$\boxed{\sum_{x_k} p(x_n, \ldots, x_k, \ldots, x_0) = p(x_n, \ldots, \cancel{x_k}, \ldots, x_0).} \tag{6}$$

This is the KCC, which is the *defining property of a stochastic process*. It says that not measuring the system at some time has the same effect as measuring it and averaging over all possible results (or measuring the system but forgetting the measurement result if you have a Bayesian attitude towards probabilities).

The KCC implies that the joint probabilities of a stochastic process form a *hierarchy*, where the k-time probabilities are contained in the n-time probabilities for $n > k$. Importantly, the so called *Daniell-Kolmogorov extension theorem* guarantees that we can also go the reverse way. If we have k-time probabilities satisfying the KCC, we can view them as marginals of n-time probabilities that also satisfy the KCC. Moreover, this holds true even in the limit $k \to \infty$. Thus, the Daniell-Kolmogorov extension theorem provides a bridge between experimental reality (where measurement statistics is always *finite*) and its theoretical description (which often uses *continuous-time* dynamics in the form of, e.g., Eq. (1)). For a theoretical physicist this sounds probably obvious, if you want to see a mathematical proof, take a look, e.g., at Ref. [4].

For the rest of this lecture it is now of outstanding importance to realize that Eq. (6) does *not come for free*. There are many relevant examples that break the KCC, for instance:

- All systems involving feedback control (just imagine the probability on the left of Eq. (6) being the probability based on the decision to apply feedback control conditioned on a measurement result at time t_k).

- Any sort of conditioning or interventions you might perform on the system break in general the KCC. Imagine, e.g., a clinical trial where you change the process by giving some (but not all) patients drugs.

- Quantum systems because quantum measurements are in general disturbing (in principle, also classical measurements could be disturbing, of course).

- In general: any sort of *active intervention* breaks KCC.

Finally, for later reference we briefly discuss the formal definition of **Markovianity** for a *stochastic process*, i.e., a process that obeys the KCC. Let

$$p(x_n|x_{n-1},\dots,x_0) \equiv \frac{p(x_n,x_{n-1},\dots,x_0)}{p(x_{n-1},\dots,x_0)} \tag{7}$$

be the conditional probability to be in state x_n at time t_n given that the system visited the states $x_{n-1},\dots,x_0$ at times $t_{n-1},\dots,t_0$. Then, the stochastic process is Markovian if

$$\boxed{p(x_n|x_{n-1},\dots,x_0) = p(x_n|x_{n-1}) \quad \forall \mathbf{x}_n, n.} \tag{8}$$

Note that this definition requires to check a very large number of conditions (in fact, infinitely many if you allow $n \to \infty$), which makes it virtually impossible to check (full) Markovianity experimentally. Nevertheless, the Markov assumption is so powerful that most text books on stochastic processes actually are text books on *Markovian* stochastic processes (which might explain the widespread confusion among scholars when it comes to non-Markovian processes). For instance, Eq. (8) implies that Eq. (4) can be neatly expressed as

$$p(x_n,\dots,x_1,x_0) = p(x_n|x_{n-1})\cdots p(x_1|x_0)p(x_0), \tag{9}$$

i.e., all what matters are the 2-time transition probabilities $p(x_{k+1}|x_k)$ (and an initial condition).

Importantly, if the KCC is broken, e.g., in case of feedback control, Eq. (8) no longer gives an adequate definition of Markovianity and the final goal of this lecture is solve this problem.

III. DISTINGUISHING CORRELATIONS AND CAUSATION

This is a wide topic, we will only touch the basics briefly. Evidently, the joint probability (4) gives us the possibility to quantify *correlations* between different 'events', for instance, between past and future values of the system state. Moreover, the system states $x \in X$ could be bipartite, i.e., the system could be composed out of two subsystems $x = (y,z) \in X = Y \times Z$, and we could quantify correlations between these two subsystems at various times.

Short digression. Since there are many ways to mathematically quantify correlations, we here introduce for later use the **mutual information (MI)** between two random variables X and Y with joint probability distribution $p(x,y)$ and marginals $p(x) = \sum_y p(x,y)$ and $p(y) = \sum_x p(x,y)$:

$$I_{X:Y} \equiv \sum_{x,y} p(x,y) \ln \frac{p(x,y)}{p(x)p(y)}. \tag{10}$$

MI is a natural way to quantify correlations as it is based on Shannon's theory of information. It can be expressed as $I_{X:Y} = H_X + H_Y - H_{XY}$ where $H_X = -\sum_x p(x) \ln p(x)$ is the Shannon entropy. MI is non-negative and zero if and only if $p(x, y) = p(x)p(y)$ is decorrelated. Moreover, MI is upper bounded as $I_{X:Y} \leq \min\{\dim X, \dim Y\}$ and the upper bound is reached for a state about which we have no prior information regarding X or Y, but once we know the value of either X or Y, the value of the other is uniquely fixed. If you feel unfamiliar with the concept of MI, it might be useful to try to prove the statements above or to look, e.g., at Ref. [1].

Now, how about *causation*? Can we infer from a stochastic process whether one event is the *cause* of another? Admittedly, this is a somewhat vague questions given that many different people might have slightly different ideas of what "causality" shall mean. For instance, Hume famously argued that the distinction between causation and correlation is merely verbal with the former being applied to mean a "directed" correlation from past to future [5]. In fact, recalling that quantum mechanics obeys time-reversal symmetry, it appears quite justified to claim that there are no causes between events in *microscopic interactions*. However, the experienced world around us does not obey time-reversal symmetry, and being able to distinguish between causation and correlation becomes vital. For instance, is smoking and cancer just correlated, or does smoking cause cancer? In the latter case, you can prolong your life when quitting smoking, in the former case you don't need to worry. Importantly, if you want to find out whether smoking causes cancer, you have to stop or start smoking to see what happens, i.e., *you have to change the process*.

We return to the case of smoking at the end of this section, but the crucial insight is that causation can be defined in a mathematically precise way if you look at how different variables react to *changes* in the process. But this implies that you break the KCC (6) because you have to actively intervene and you can no longer just passively observe the process. This way of thinking about causality was made precise by Judea Pearl and others [2]. Is that the only way? I don't know, but I am not aware of any other precise and non-redundant mathematical definition of causation. Below, I copy one example from Ref. [1] to make this reasoning precise. A more general mathematical theory will be developed later.

Example. We consider three binary random variables S, B and C with values s, b and c. On a given day S describes whether the sun is shining ($s = 1$) or not ($s = 0$), B describes whether the number of sunburns is high ($b = 1$) or low ($b = 0$) and C describes whether the number of ice cream sales is high ($c = 1$) or low ($c = 0$). We assume the notion of 'high'/'low' to be

chosen according to some reasonable treshhold. We further set the conditional probabilities to be $p(b=1|s=1)=p(c=1|s=1)=p(b=0|s=0)=p(c=0|s=0)=\lambda$ with $1/2<\lambda\leq 1$, e.g., $\lambda=1$ implies that the number of sunburns is always high if the sun is shining. By conservation of probability, $p(b=0|s=1)=p(c=0|s=1)=p(b=1|s=0)=p(c=1|s=0)=1-\lambda$. Furthermore, we assume the probability for a sunny day to be $p(s=1)=1/2$, which implies that the sun is not shining with the same probability $p(s=0)=1/2$.

Now, we first look at the correlations between B and C as quantified by their MI:

$$I_{B:C}\equiv\sum_{b,c}p(b,c)\ln\frac{p(b,c)}{p(b)p(c)}=\ln 2-H_{\rm bin}(2\lambda-2\lambda^2). \tag{11}$$

Here, $H_{\rm bin}(p)\equiv -p\ln p-(1-p)\ln(1-p)$ denotes the *binary* Shannon entropy and we computed Eq. (11) by using $p(b,c)=\sum_s p(b,c,s)=\sum_s p(b|s)p(c|s)p(s)$. We see that $I_{B:C}=0$ (no correlations) implies $\lambda=1/2$ and $I_{B:C}=\ln 2$ (maximal correlations) implies $\lambda=1$. Thus, B and C are in general correlated as expected

Next, we compute the correlations between S and B and find $I_{B:S}=\ln 2-H_{\rm bin}(\lambda)$, i.e., S and B are maximally correlated (uncorrelated) for $\lambda=1$ ($\lambda=1/2$) as before. However, we strongly believe that S is the cause of B (and also of C), i.e., the shining sun triggers sun burns (and ice cream sales), but we do not believe that B is the cause of C or vice versa. How can we make this intuition rigorous?

Assume that we have an external mechanism, which can change whether the sun is shining or not (which seems unrealistic at first sight, but we come back to it below). We therefore introduce an additional *intervention variable* I_S (not to be mixed up with the MI), which labels the following three actions i: Do nothing, i.e., leave the sun as it is ($i=$ idle), make the sun shining ($i=1$), or block sun shine ($i=0$). Now, consider the conditional probability $p(b|s,i)$ for sun burns given sunshine s and intervention i. For $i=$ idle we set $p(b|s,\text{idle})=p(b|s)$ as defined above. Furthermore, we assume $p(b|s,i=1)=\lambda$ and $p(b|s,i=0)=1-\lambda$ *independent* of s because $i\in\{0,1\}$ overwrites the natural value of s to be identical to i. Then, we find the MI between B and I_S to be

$$I_{B:I_S}=\sum_{b,i}p(b,i)\ln\frac{p(b,i)}{p(b)p(i)}=H_{\rm bin}[p(b)]-[1-p(i=\text{idle})]H_{\rm bin}(\lambda)-p(i=\text{idle})\ln 2, \tag{12}$$

where the marginal probability $p(i)$ that we perform a certain intervention is assumed to be controllable in an experiment. We now define that *S is a **cause** of B if there are correlations between I_S and B*. From Eq. (12) we infer that there are no correlations between B and I_S, i.e., S is *not* the cause of B, if one of the following two cases happen: either $p(i=\text{idle})=1$, which corresponds to the case that we do not perform any intervention and hence, cannot test for causality, or $\lambda=1/2$,

which implies that there are no correlations between B and S at first place. Furthermore, we find in general $p(b,s) \neq \sum_i p(b,s,i)$, where $p(b,s) = p(b|s)p(s)$ is the joint probability from the beginning obtained *without* interventions and $p(b,s,i) = p(b|s,i)p(s)p(i)$ is the joint probability with interventions. Thus, the KCC is broken in general unless $p(i = \text{idle}) = 1$ or $p(i = 0) = p(i = 1)$. Similarly, we could also replace B by C above and find that sunshine *causes* a high number of ice cream sales.

Finally, let us return to our assumption that we can change the sunlight by an external intervention. Indeed, such a mechanism is not easy to construct for a human. But to distinguish causation and correlation, it is not necessary that humans perform the intervention: it could be also done by nature, for instance, due to a solar eclipse. Important is only that we can fix the intervention variable independent of the other variables in the model.

A. Does smoking cause lung cancer?

We are now (for good reasons) convinced that it does so, but this question caused a serious scientific debate for around 15 years after World War 2. Why that? At that time scientists detected a strong correlation between smoking and lung cancer, but could they use this evidence to claim a causal link between the two? Consider the alternative explanation: *There exists a special gene that makes you both love cigarettes and prone to lung cancer.* Can you rule out this explanation based on the detected correlation between smoking and lung cancer? Not very easily and this caused a big crisis among mathematicians and statisticians when they realized the shortcomings of their method.

How could you detect that smoking causes lung cancer? The simplest direct way would be to randomly select non-smokers and to force them to smoke 2 packs of cigarettes a day for 30 years. Clearly, this way was not considered to be ethical, not to mention the problem that waiting for 30 years for the results is quite unsatisfactory. At the end, the debate was somewhat settled in a rather indirect way by Jerome Cornfield *et al.* in 1959: They argued that, if such a gene existed, it must have an enormously strong influence on people to explain the correlations in the data. That a complex social phenomenon such as smoking is determined by such a single gene variation was considered to be absurd by most biologists.

A more detailed history about the smoking-cancer debate is recollected in Ref. [6]. The somewhat saddest (and most interesting) point is that this debate is the first clear example of organized denialism where powerful companies deliberately deceive the public about scientific facts. The

tobacco industry founded and funded research committees, institutes, think tanks, and payed scientists and journalists to obscure the truth, and they did this very successfully: the federal non-smoking act ("Bundesnichtraucherschutzgesetz") in Germany was published only in 2007!

Today, we are facing a much more existential threat from organized denialism: climate change! Readers interested in finding out how the public was and is deceived about smoking or climate change are strongly recommend to read Ref. [7]. Alright, let's get back to our ivory tower...

IV. MODELLING INTERVENTIONS: STATES AND MAPS

After the necessity for more general interventions instead of only passive measurements has become (hopefully) clear, we develop a mathematical framework for that. We start with the classical case first and then turn to the quantum case. Perhaps surprisingly, both share a lot of similarities, and in my view the quantum case is actually more elegant (and also needed for the rest of this lecture). Moreover, here we only focus on interventions happening at a *single time*. How to concatenate them to describe a *process* is explained in later sections.

A. Classical case

The state of a classical system can be conveniently described by a probability vector $\mathbf{p}$ with entries $p(x_1), p(x_2), \ldots, p(x_d)$, where $d = \dim X$ is the dimension of this vector, i.e., the number of distinct system states that we can distinguish. For instance, a 'pure' state is given by $p(x_i) = \delta_{ij}$ (Kronecker delta) for some $j \in \{1, \ldots, d\}$, the maximally mixed state is $p(x_i) = 1/d$ for all i.

This state changes upon making an *intervention*, which is a broad terminology comprising measurements with or without error, feedback control operations, state (re)preparations, noise additions, etc. Clearly, the change of a vector is conveniently described by a matrix M and we denote the state after the intervention as $\tilde{\mathbf{p}}' = M\mathbf{p}$. Working with a matrix guarantees the important property of (convex) linearity: Suppose we have a state $\mathbf{p} = \lambda\mathbf{p}_1 + (1-\lambda)\mathbf{p}_2$ with $\lambda \in [0, 1]$, which is an ensemble of two different states $\mathbf{p}_1$ and $\mathbf{p}_2$ (resulting, e.g., from different initial state preparations), then $M\mathbf{p} = \lambda M\mathbf{p}_1 + (1-\lambda)M\mathbf{p}_2$, i.e., we can compute the effect of the intervention on the different ensemble members separately to compute the joint effect.

As a warm-up example, we consider a non-disturbing measurement. If this measurement is error-free, then we can write the matrix $M(x_i)$ corresponding to finding the system in state x_i as

$$[M(x_i)]_{mn} = \delta_{mi}\delta_{ni} \quad \text{or in Dirac notation (which we use from now on)} \quad M(x_i) = |x_i\rangle\langle x_i|. \tag{13}$$

It transforms the system state according to $|\tilde{\mathbf{p}}'(x_i)\rangle = M(x_i)|\mathbf{p}\rangle = p(x_i)|x_i\rangle$, i.e., $|\tilde{\mathbf{p}}'(x_i)\rangle$ has zeros everywhere except at position i, where it contains the probability to find the system in state x_i. Notice that we put an additional "tilde" $\sim$ on top of the output state to indicate that $|\tilde{\mathbf{p}}'(x_i)\rangle$ is *no longer normalized*: its components do not sum up to one! Instead, its components sum up to the probability $p(x_i) \leq 1$ to measure x_i. This formulation might seem awkward at first sight but will turn out to be advantageous. Finally, repeating the experiment many times gives in general rise to different measurement outcomes, so $M(x_i)$ is just a member of a set of matrices $\{M(x_1), \ldots, M(x_d)\}$.

How can we generalize this situation to measurements with errors? To this end, it is convenient to introduce another random variable r to denote the measurement *result*. Then, let $p(r|x_i)$ be the conditional probability to obtain result r given that the system is in state x_i. For instance, for an error-free measurement $p(r|x_i) = \delta_{r,i}$, but in general the conditional probability only needs to obey $p(r|x_i) \geq 0$ for all r and x_i and $\sum_r p(r|x_i) = 1$. An imperfect measurement is then described by a set of matrices $\{M(r)\}$ with $M(r) = \sum_i p(r|x_i)|x_i\rangle\langle x_i|$, which transform the system state as

$$|\tilde{\mathbf{p}}'(r)\rangle = M(r)|\mathbf{p}\rangle = \sum_i p(r|x_i)p(x_i)|x_i\rangle. \tag{14}$$

Again, $|\tilde{\mathbf{p}}'(r)\rangle$ is not normalized, but the sum of its components equals the probability $p'(r) \equiv \sum_i p(r|x_i)p(x_i)$ to obtain result r. We can normalize $|\tilde{\mathbf{p}}'(r)\rangle$ by dividing it by $p'(r)$. The components of this normalized vectors are then the conditional probability to find the system in state x_i given that we measured result r:

$$p'(x_i|r) = \frac{\langle x_i|\tilde{\mathbf{p}}'(r)\rangle}{p'(r)} = \frac{p(r|x_i)p(x_i)}{p'(r)}. \tag{15}$$

The reader should recognize this equation as **Bayes' rule**.

So far the examples above involved only *diagonal* matrices. The general picture is the following (an example follows afterwards). Any classical intervention is characterized by a set of matrices $\{M(r)\}$ with at least one (and in principle infinitely many) elements. Each element has to satisfy the following property:

- Positivity: $\langle x_i|M(r)|x_j\rangle \geq 0$ for all x_i, x_j and r. This property ensures that all state vectors remain positive, i.e., negative probabilities cannot occur.

Moreover, depending on the state $\mathbf{p}$ prior to the intervention, each $M(r)$ can be realized with a different probability given by $p'(r) = \sum_i \langle x_i|\tilde{\mathbf{p}}'(r)\rangle$. However, in every repetition of the experiment one $M(r)$ must be realized. This translates to the following property:

- Normalization: Let $M \equiv \sum_r M(r)$ describe the average affect of the intervention. Then, $\sum_i \langle x_i|M|x_j\rangle = 1$ for all x_j.

Since the positivity requirements implies $\langle x_i|M|x_j\rangle \geq 0$, we notice that M is a *stochastic matrix*: it is the most general transformation between probability vectors. Each $M(r)$ can be called a *substochastic matrix*: it maps probabilities to non-normalized "probabilities" whose norm (= sum of its components) equals the probability for $M(r)$ to occur.

Readers who found the last points very abstract are invited to explicitly confirm them for the case of a measurement with error as introduced above (Bayes' rule). Moreover, one easily confirms in that case that $\sum_r |\tilde{\mathbf{p}}'(r)\rangle = M|\mathbf{p}\rangle = |\mathbf{p}\rangle$, i.e., on average the state vector does not change. Put differently, Bayes' rule describes the most general *non-disturbing* classical measurement.

Example for a disturbing classical measurement. Suppose you wonder whether you have a gas leak in your apartment or not. Let $x = 0$ ($x = 1$) describe the state of your apartment with no (with) gas inside. One way to find out whether you have gas inside your apartment, is to wait outside and to ask your "best" friend to go and smoke a cigarette in your apartment. Your measurement outcomes will be $r = 0$ or $r = 1$, depending on the question whether your friend survives or not, respectively. But whatever the result is, you will for sure have no gas in your apartment after the measurement. With respect to the ordered basis $(|x = 0\rangle, |x = 1\rangle) = (\text{no gas}, \text{gas})$ we then find the following state transformation matrices:

$$M(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad M = M(0) + M(1) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}. \tag{16}$$

B. Quantum case

In quantum mechanics the state of a system is described by a density matrix ρ and changes in the system state can be conveniently described using again linear maps $\mathcal{C} : \rho \mapsto \rho' = \mathcal{C}\rho$. Since ρ itself is a matrix (or 'operator'), such linear maps $\mathcal{C}$ are often called *superoperators*. In essence, superoperators are just big matrices themselves. To see this, consider a quantum system with Hilbert space dimension $d = \dim \mathcal{H}$. Then, ρ is a $d \times d$ matrix with complex entries, but since matrices form a vector space, we can represent ρ alternatively by a complex vector with d^2 many entries. On that vector space $\mathcal{C}$ can be represented by an ordinary $d^2 \times d^2$ matrix (which is in fact typically done in numerical applications).

Sparing examples for later on, we start with the general theory, which much parallels what we found for classical systems (with the difference that everything happens "one dimension higher"). Hence, an intervention in quantum mechanics is described by a set $\{\mathcal{C}(r)\}$ of superoperators and we denote the average effect by $\mathcal{C} \equiv \sum_r \mathcal{C}(r)$. Moreover, we like to ensure that any intervention preserves the characteristic properties of the density matrix, i.e., we want that the output states $\tilde{\rho}'(r) = \mathcal{C}(r)\rho$ describe a legitimate set of physical states for all legitimate input states ρ. This gives rise to essentially the same requirements as above:

- Positivity: If $\rho \geq 0$, we want that $\mathcal{C}(r)\rho \geq 0$ too. In fact, it turns out that this notion is not strong enough. Consider a bipartite quantum system $\mathcal{H}_X \otimes \mathcal{H}_Y$ with state ρ_{XY} and suppose $\mathcal{C}(r)$ is a map acting only on states in X. Then, we want for any input state $\rho_{XY} \geq 0$ that $[\mathcal{C}(r) \otimes \mathcal{I}_Y]\rho_{XY} \geq 0$ where $\mathcal{I}_Y$ denotes the identity operation on Y. This is called **complete positivity**.[2]

- Normalization: If $\mathrm{tr}\{\rho\} = 1$, we want that $\mathrm{tr}\{\mathcal{C}\rho\} = 1$ too (this property is completely analogous to the classical case).

As before, while probability is preserved on average, $\mathrm{tr}\{\mathcal{C}\rho\} = 1$, every single element $\mathcal{C}(r)$ is applied with a certain probability $p(r) \equiv \mathrm{tr}\{\mathcal{C}(r)\rho\} \leq 1$. Therefore, the state $\tilde{\rho}'(r) = \mathcal{C}(r)\rho$ has in general not unit trace and we again use a tilde $\sim$ to denote this.

To introduce some standard terminology in the field, we call the $\mathcal{C}(r)$ *completely positive (CP) maps* and $\mathcal{C} = \sum_r \mathcal{C}(r)$ a *completely positive and trace-preserving (CPTP) map.* A set of CP maps $\{\mathcal{C}(r)\}$ that adds up to a CPTP map is called an **instrument**. Moreover, there are two very important representation theorems to mathematically characterize CP and CPTP maps:

Operator-sum representation. *A map $\mathcal{C}$ is CPTP if and only if it can be written as*

$$\mathcal{C}\rho = \sum_\alpha K_\alpha \rho K_\alpha^\dagger \tag{17}$$

for an arbitrary set of operators K_α (often called "Kraus operators") satisfying $\sum_\alpha K_\alpha^\dagger K_\alpha = I$ (with I the identity). The CP maps $\mathcal{C}(r)$ can be also written as in Eq. (17) for some set of operators K_α satisfying $\sum_\alpha K_\alpha^\dagger K_\alpha \leq I$ (with I the identity).

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[2] The reader can check that positivity implies complete positivity in the classical case, but for quantum systems this is in general not true in presence of entanglement.

Unitary (Stinespring) dilation theorem. *A set of maps $\mathcal{C}(r)$ forms an instrument if and only if there exists a d^2-dimensional "ancilla" Hilbert space $\mathcal{H}_A$, a unitary U acting on the system and ancilla space, a pure ancilla state $|\phi\rangle_A \in \mathcal{H}_A$ and a set of projectors $\{\Pi_A(r)\}$ acting on the ancilla space such that*

$$\mathcal{C}(r)\rho_S = \mathrm{tr}_A\{\Pi_A(r)U_{SA}(\rho_S \otimes |\phi\rangle\langle\phi|_A)U_{SA}^\dagger\} \tag{18}$$

for all r. In particular, a map $\mathcal{C}$ is CPTP if and only if it can be written as

$$\mathcal{C}\rho_S = \mathrm{tr}_A\{U_{SA}(\rho_S \otimes |\phi\rangle\langle\phi|_A)U_{SA}^\dagger\}. \tag{19}$$

In particular the last representation theorem implies that any instrument can be generated by unitary evolution (Schrödinger dynamics) and standard projective measurements in a suitable larger system-ancilla space. A very similar theorem holds in the classical case with one important difference: owing to the absence of entanglement, the initial state of the ancilla can in general not be chosen to be pure but must be mixed [8].

This concludes our general exposition of single-time interventions in quantum systems. All what we have done was to introduce terminology and notation to describe state changes a system can experience in a lab when externally manipulated by some agent or experimenter. The examples below make the abstract exposition above hopefully clearer. If not, feel free to replace density matrices and superoperators by probability vectors and matrices as introduced in the classical case. All what matters is the idea that there is a mathematical formalism to describe state changes, which disturb the system and thus violate the Kolmogorov consistency condition (KCC) because in general we have $|\mathbf{p}'\rangle = M|\mathbf{p}\rangle \neq |\mathbf{p}\rangle$ and $\rho' = \mathcal{C}\rho \neq \rho$ in both, the classical and quantum case.

Exercise. Find either an explicit operator-sum representation or unitary dilation to show that the following sets of maps are instruments:

- Unitary time-evolution: The set $\{\mathcal{C}(r)\}$ has only a single element $\mathcal{C}$ and define $\mathcal{C}\rho \equiv U\rho U^\dagger$ for some unitary operator U satisfying $UU^\dagger = U^\dagger U = I$.

- Projective measurement: Consider an observable $R = \sum_r \lambda_r \Pi_r$ with eigenvalues λ_r and corresponding projectors Π_r. Now, set $\mathcal{C}(r)\rho \equiv \Pi_r\rho\Pi_r$ such that $\mathcal{C}\rho = \sum_r \Pi_r\rho\Pi_r$. Verify that $p(r) = \mathrm{tr}\{\mathcal{C}(r)\rho\}$ is identical to Born's rule.

V. QUANTUM STOCHASTIC PROCESSES

Recall the basic idea of a stochastic process: there was a multi-time object $p(x_n, t_n; \ldots; x_1, t_1; x_0, t_0)$ that described the probability to obtain measurement results $x_0, x_1, \ldots, x_n$ at times $t_0 < t_1 < \cdots < t_n$. Next, recall our insight about causal models: if we want to describe feedback control or quantum systems, we have to overcome the idea of non-disturbing interventions (i.e., the KCC) and we need some multi-time object $P[\mathcal{C}_n(r_n), \ldots, \mathcal{C}_1(r_1), \mathcal{C}_0(r_0)]$ that describes the probability to obtain results $r_0, r_1, \ldots, r_n$ given that we applied the interventions $\mathcal{C}_0(r_0), \mathcal{C}(r_1), \ldots, \mathcal{C}(r_n)$. Can we construct such an object in a well-defined manner? It turns out we can and we will construct it explicitly now.

To this end we assume that our system of interest S is coupled to an environment E. The total state of system and environment at time t is $\rho_{SE}(t)$. The total Hamiltonian of S and E is written as $H = H_S + H_E + V$, where H_S (H_E) is the Hamiltonian of the system (environment) alone and V describes their interaction. *In absence* of any external interventions, the total state evolves from time s to time t according to

$$\rho_{SE}(t) = U_{t,s}\rho_{SE}(s)U_{t,s}^\dagger \equiv \mathcal{U}(t,s)\rho_{SE}(s) \quad \text{with} \quad U_{t,s} = e^{-iH(t-s)} \quad (\hbar \equiv 1). \tag{20}$$

Now, suppose we perform interventions at times $t_0 < t_1 < \cdots < t_n$ using instruments $\{\mathcal{C}_0(r_0)\}, \{\mathcal{C}_1(r_1)\}, \ldots, \{\mathcal{C}_n(r_n)\}$ (note that the choice of instruments can change from time to time). The system-environment state conditioned on receiving results $\mathbf{r}_n = (r_n, \ldots, r_1, r_0)$ at time t_n right *after* the last intervention is

$$\tilde{\rho}'_{SE}(t_n|\mathbf{r}_n) = \mathcal{C}_n(r_n)\mathcal{U}(t_n, t_{n-1}) \cdots \mathcal{C}_1(r_1)\mathcal{U}(t_1, t_0)\mathcal{C}_0(r_0)\rho_{SE}(t_0), \tag{21}$$

where $\rho_{SE}(t_0)$ describes the initial state prior to the first intervention. Moreover, we suppressed any tensor products with the identity operation in the notation, i.e., $\mathcal{C}_k(r_k) \otimes \mathcal{I}_E = \mathcal{C}_k(r_k)$. Since the environmental state in inaccessible and we are only interested in the system state, we trace over the environment:

$$\begin{aligned} \tilde{\rho}'_S(t_n|\mathbf{r}_n) &= \mathrm{tr}_E \left\{\mathcal{C}_n(r_n)\mathcal{U}(t_n, t_{n-1}) \cdots \mathcal{C}_1(r_1)\mathcal{U}(t_1, t_0)\mathcal{C}_0(r_0)\rho_{SE}(t_0)\right\} \\ &\equiv \mathfrak{T}[\mathcal{C}_n(r_n), \ldots, \mathcal{C}_1(r_1), \mathcal{C}_0(r_0)]. \end{aligned} \tag{22}$$

Here, we have introduced the **process tensor** $\mathfrak{T}[\mathcal{C}_n(r_n), \ldots, \mathcal{C}_1(r_1), \mathcal{C}_0(r_0)]$. It is a formal object that describes the response of an open quantum system to any sequence of interventions. It is the most abstract object we will meet in this lecture and it is certainly extremely hard to compute the process tensor in many practical applications. However, we will not attempt to compute $\mathfrak{T}$

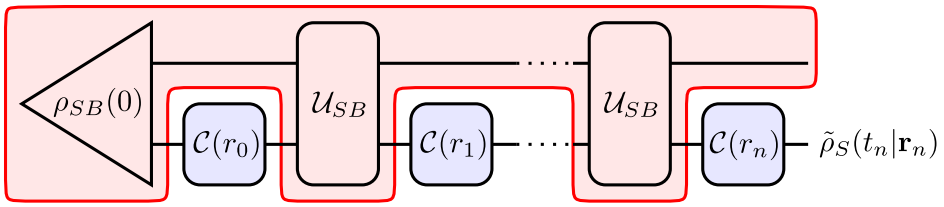

Figure 1. Graphical representation of the process tensor with time running from left to right.

explicitly here; what matters for us is that it is a universal tool that allows us to address many complicated question within a unified framework. So let's try to find out some general properties of it:

1. First, a graphical representation of the process tensor is given in Fig. 1. The process tensor comprises the effect of everything colored in red, whereas the blue boxes represent the interventions that an experimenter is free to choose (in contrast, everything colored in red is not under (detailed) control of the experimenter). Note that the red object looks like a *comb*. Therefore, the process tensor is also often called a **quantum comb** [9].

2. The process tensor really is a *tensor*, i.e., an object acting multi-linearly on its entries. Indeed, you can easily confirm that

$$\mathfrak{T}[\mathcal{A}_n, \ldots, a\mathcal{A}_k + b\mathcal{A}'_k, \ldots, \mathcal{A}_0] = a\mathfrak{T}[\mathcal{A}_n, \ldots, \mathcal{A}_k, \ldots, \mathcal{A}_0] + b\mathfrak{T}[\mathcal{A}_n, \ldots, \mathcal{A}'_k, \ldots, \mathcal{A}_0] \tag{23}$$

   for any $k \in \{0, \ldots, n\}$, any maps $\mathcal{A}_k$ and $\mathcal{A}'_k$ and any complex numbers $a, b \in \mathbb{C}$.

   If $\mathcal{H}_S$ denotes the Hilbert space of the system, then we write $\rho \in L(\mathcal{H}_S)$ with $L(\mathcal{H}_S)$ the space of linear maps acting on $\mathcal{H}_S$. Following this notation, a superoperator lives in $L(L(\mathcal{H}_S))$ (the space of linear maps acting on linear maps over $\mathcal{H}_S$), and consequently the process tensor describes the map:

$$\mathfrak{T} : \underbrace{L(L(\mathcal{H}_S)) \otimes \cdots \otimes L(L(\mathcal{H}_S))}_{n+1 \text{ times}} \to L(\mathcal{H}_S). \tag{24}$$

   That is, $\mathfrak{T}$ is a *"super-superoperator"*.

3. Since $\mathfrak{T}$ acts linearly on each of its entries, it suffices to know its action for one particular set of basis interventions to know the action of $\mathfrak{T}$ for any intervention. Recall that $L(L(\mathcal{H}_S))$ is a vector space and thus every superoperator $\mathcal{A} = \sum_{\alpha,\beta} c_{\alpha\beta}\mathcal{B}_{\alpha\beta}$ can be written as a linear combination of basis superoperators $\mathcal{B}_{\alpha\beta}$ and complex coefficients $c_{\alpha\beta} \in \mathbb{C}$. A particularly

convenient basis $\{\mathcal{B}_{\alpha\beta}\}$ is given by superoperators of the form

$$\mathcal{B}_{\alpha\beta}\rho_S \equiv \sigma_S^{(\alpha)}\mathrm{tr}\{\Pi_\beta \rho_S\}. \tag{25}$$

Here, $\{\Pi_\beta\}$ is an "informationally complete" set of d^2 many projectors[3] and σ_α is a linearly independent set of d^2 many quantum states (density matrices). The operational meaning of Eq. (25) is thus to measure the quantum system with respect to some projector Π_β and to prepare it afterwards in the new state σ_α. Any action of the process tensor can then be inferred from its basis elements

$$\mathfrak{T}\left[\mathcal{B}_{\alpha_n\beta_n}, \ldots, \mathcal{B}_{\alpha_1\beta_1}, \mathcal{B}_{\alpha_0\beta_0}\right] \tag{26}$$

and suitable linear combination. Note that there are $d^{4(n+1)}$ many such basis elements!

4. Using the basis above, we see that the sequence of control operations $\mathcal{C}_n(r_n), \ldots, \mathcal{C}_1(r_1), \mathcal{C}_0(r_0)$ is associated to the vector

$$\sum_{\alpha_n,\beta_n} c_{\alpha_n\beta_n}\mathcal{B}_{\alpha_n\beta_n} \otimes \cdots \otimes \sum_{\alpha_1,\beta_1} c_{\alpha_1\beta_1}\mathcal{B}_{\alpha_1\beta_1} \otimes \sum_{\alpha_0,\beta_0} c_{\alpha_0\beta_0}\mathcal{B}_{\alpha\beta} \tag{27}$$

living in the $(n+1)$-fold tensor product of $L(L(\mathcal{H}_S))$. We now recognize that choosing at each time t_k an independent instrument $\{\mathcal{C}_k(r_k)\}$ just represents one possible intervention out of a much larger class. In general, any intervention of the form

$$\sum_{\alpha_n,\beta_n} \cdots \sum_{\alpha_1,\beta_1} \sum_{\alpha_0,\beta_0} c_{\alpha_n\beta_n,\ldots,\alpha_1\beta_1,\alpha_0\beta_0}\mathcal{B}_{\alpha_n\beta_n} \otimes \cdots \otimes \mathcal{B}_{\alpha_1\beta_1} \otimes \mathcal{B}_{\alpha\beta} \tag{28}$$

is conceivable, where the coefficients $c_{\alpha_n\beta_n,\ldots,\alpha_1\beta_1,\alpha_0\beta_0}$ need not factorize as in Eq. (27). This allows to implement a much larger class of control operations—in fact, it allows to implement the *most general state transformation permitted by the laws of quantum mechanics and the requirement of causality* (meaning that we have an ordered set of times $t_0 < t_1 < \cdots < t_n$). Thus, the formalism above is also known as a **quantum causal model** (see Refs. [3] and references therein). Nevertheless, we remark that *the response of the system to any whatsoever complicated intervention is contained in the basis elements (26) describing simple independent measure-and-prepare interventions.* This is a very important statement and follows from the linearity of any stochastic theory.

[3] Informationally complete here means that the set $\{\Pi_\beta\}$ is a basis for $L(\mathcal{H}_S)$. For instance, for a qubit an informationally complete basis is given by the two projectors on the eigenspaces of σ_z plus one projector on any one eigenspace of both σ_x and σ_y.

5. Finally, notice that the identity operation $\mathcal{I}$ is a legitimate intervention too ("do nothing"). Thus, the process tensor defined on the set of times $\{t_0, t_1, \ldots, t_n\}$ *contains* the process tensors defined on any subset of times. This generalizes the KCC (6) for arbitrary interventions and allows to prove a generalized extension theorem [10].

To conclude, the process tensor encodes the response of an open system for all possible interventions that one can apply to it, and the probability for a particular intervention is

$$P(\mathbf{r}_n) \equiv P[\mathcal{C}_n(r_n), \ldots, \mathcal{C}_1(r_1), \mathcal{C}_0(r_0)] = \mathrm{tr}_S\{\mathfrak{T}[\mathcal{C}_n(r_n), \ldots, \mathcal{C}_1(r_1), \mathcal{C}_0(r_0)]\}. \tag{29}$$

This suggests to identify the process tensor formalism with the notion of a **quantum stochastic process**.

VI. OPERATIONAL MARKOV CONDITION

The formalism developed above is valid for Markovian or non-Markovian dynamics. The question is now: given the process tensor, how can we decide whether the process is Markovian?

To this end, we introduce the notion of a **causal break**. A causal break is a specific intervention $\mathcal{B}$, which ensures that the output state $\rho'_S = \mathcal{B}\rho_S$ is *independent* of the input state ρ_S. This guarantees that the *system state* ρ'_S carries no memory about *its past*. Thus, if after a causal break the future evolution of the system depends on past interventions, this can only be because the *environment* retained a memory about past interventions in a way that influences the system dynamics.

Mathematically, it might not be surprising that causal breaks can be written in the form of Eq. (25): whatever the input state ρ_S was, the output state is $\sigma_S^{(\alpha)}$. Moreoever, recall that the final state of the process in Eq. (22) is not normalized, but occurs with probability (29). We denote the corresponding normalized state as

$$\rho_n[\mathcal{C}_n(r_n), \ldots, \mathcal{C}_1(r_1), \mathcal{C}_0(r_0)] \equiv \frac{\tilde{\rho}'_S(t_n|\mathbf{r}_n)}{P(\mathbf{r}_n)} = \frac{\mathfrak{T}[\mathcal{C}_n(r_n), \ldots, \mathcal{C}_1(r_1), \mathcal{C}_0(r_0)]}{\mathrm{tr}_S\{\mathfrak{T}[\mathcal{C}_n(r_n), \ldots, \mathcal{C}_1(r_1), \mathcal{C}_0(r_0)]\}}. \tag{30}$$

We can now define a quantum Markov process as first done in Ref. [11] (we remark that this definition reduces to the causal Markov condition for a classical causal model [2]):

Quantum Markov process. *A quantum stochastic process is Markovian if the normalized system state at time t_ℓ after a causal break at time $t_k < t_\ell$ depends only on the input state $\sigma_S^{(\alpha_k)}$ for any*

set of previous interventions $\mathcal{C}(r_{k-1})$, ..., $\mathcal{C}(r_0)$ *and for all* $t_k < t_\ell$. *In equations,*

$$\boxed{\rho_\ell[\mathcal{B}_{\alpha_k\beta_k}, \mathcal{C}(r_{k-1}), \ldots, \mathcal{C}(r_0)] = \rho_\ell[\sigma_S^{(\alpha_k)}].} \tag{31}$$

Now, Markov processes are popular because they imply strong theoretical simplifications. Instead of dealing with the complicated process tensor of Eq. (22), it turns out that the process tensor "factorizes" and can be written for any sequence $\mathcal{C}_n, \ldots, \mathcal{C}_1, \mathcal{C}_0$ of superoperators as

$$\boxed{\mathfrak{T}[\mathcal{C}_n, \ldots, \mathcal{C}_1, \mathcal{C}_0] = \mathcal{C}_n \mathcal{E}(t_n, t_{n-1}) \cdots \mathcal{C}_1 \mathcal{E}(t_1, t_0) \mathcal{C}_0 \rho_S(t_0).} \tag{32}$$

Here, $\rho_S(t_0)$ is the initial system state and $\{\mathcal{E}(t_k, t_{k-1}) | k \in \{1, \ldots, n\}\}$ is a set of CPTP maps, which propagate the system state forward in time from t_{k-1} to t_k and which are *independent* of the applied interventions. In the theory of open quantum systems, these maps are called *dynamical maps*; for a classical stochastic system one would call them *transition matrices*. We will not derive Eq. (32) here, but hope that the reader finds it intuitive that the time evolution of a Markov process is described by a set of maps or propagators independent of the applied interventions (the eager reader with some background in open quantum system theory is, of course, invited to attempt a proof).

Now, consider the following simple example involving time-delayed feedback. At time t_0 we apply an intervention $\mathcal{C}_0(r_0)$ (some measurement) and receive result r_0, then at time t_1 we apply a causal break $\mathcal{B}_{\alpha_1\beta_1}$, and at time t_2 we apply an intervention $\mathcal{C}_2(r_0)$, which depends on the first measurement result (that's the time-delayed feedback control). Does the resulting normalized system state depend on $\mathcal{C}_0(r_0)$? A quick calculation reveals

$$\begin{aligned}
\rho_2[\mathcal{C}_2(r_0), \mathcal{B}_{\alpha_1\beta_1}, \mathcal{C}_0(r_0)] &= \frac{\mathcal{C}_2(r_0)\mathcal{E}(t_2,t_1)\mathcal{B}_{\alpha_1\beta_1}\mathcal{E}(t_1,t_0)\mathcal{C}_0(r_0)\rho_S(t_0)}{\mathrm{tr}_S\{\mathcal{C}_2(r_0)\mathcal{E}(t_2,t_1)\mathcal{B}_{\alpha_1\beta_1}\mathcal{E}(t_1,t_0)\mathcal{C}_0(r_0)\rho_S(t_0)\}} \\
&= \frac{\mathcal{C}_2(r_0)\mathcal{E}(t_2,t_1)\sigma_S^{(\alpha_1)}\mathrm{tr}_S\{\Pi_{\beta_1}\mathcal{E}(t_1,t_0)\mathcal{C}_0(r_0)\rho_S(t_0)\}}{\mathrm{tr}_S\{\mathcal{C}_2(r_0)\mathcal{E}(t_2,t_1)\sigma_S^{(\alpha_1)}\}\mathrm{tr}_S\{\Pi_{\beta_1}\mathcal{E}(t_1,t_0)\mathcal{C}_0(r_0)\rho_S(t_0)\}} \\
&= \frac{\mathcal{C}_2(r_0)\mathcal{E}(t_2,t_1)\sigma_S^{(\alpha_1)}}{\mathrm{tr}_S\{\mathcal{C}_2(r_0)\mathcal{E}(t_2,t_1)\sigma_S^{(\alpha_1)}\}} = \rho_2[\mathcal{C}_2(r_0), \sigma_S^{(\alpha_1)}],
\end{aligned} \tag{33}$$

i.e., it does *not*. This appears counterintuitive as one could (rightfully) claim that, if we hadn't received result r_0 in the first measurement, we wouldn't apply the control operation $\mathcal{C}_2(r_0)$ at time t_2. However, notice that no information about the earlier system state at time t_0 is transmitted *through the environment*: the dynamical map $\mathcal{E}(t_2, t_1)$ does not depend on $\mathcal{C}_0(r_0)$. It is only because we (or the external agent) decide to retain some memory about r_0 reflected in the intervention $\mathcal{C}_2(r_0)$, which is something *extrinsic* to the process tensor.

This might become even more clear by recalling point 4. from the last section. The experimenter could first decide to infer all basis elements (26) from decorrelated causal breaks (no feedback control involved), and afterwards they could compute the response of the system to any feedback control law by suitable linear combination. That is, the decision of the experimenter to keep some memory and to apply a "non-Markovian" control strategy is completely extrinsic to the process itself. The question whether the process is Markovian (or not) is determined by the nature of the environment and can be decided for any basis of interventions based on Eq. (31).

Finally, another perspective is offered by imagining that the experimenter simply decides from the beginning on to apply a causal break $\mathcal{B}_{\alpha_1\beta_1}$ at time t_1 and an intervention $\mathcal{C}_2(r_0)$ at time t_2 (with the value of r_0 decided independently). Does the resulting final state $\rho_2[\mathcal{C}_2(r_0), \mathcal{B}_{\alpha_1\beta_1}, \mathcal{I}]$ differ from the state $\rho_2[\mathcal{C}_2(r_0), \mathcal{B}_{\alpha_1\beta_1}, \mathcal{C}_0(r_0)]$ if we had applied a measurement and feedback loop? It does *not* because the environment is unable to transmit any memory about the system state prior to the causal break.

However, a naive application of the Markov condition (8) for a classical stochastic process would typically imply that $p(x_2|x_1, x_0) \neq p(x_2|x_1)$, i.e., the process looks non-Markovian from that point of view. This problem is caused by the fact that the language of a classical stochastic process is not suitable to distinguish between properties *intrinsic to the process* and *external influences*. And in fact, if there are no disturbing interventions, this distinction is superfluous: a measurement of the system state x at time t just reveals some preexisting property and does not change the process. To summarize, Eq. (8) checks a general condition on correlations in time *independent* of their origin, whereas Eq. (31) checks for correlations in time *caused* by the environment.

Naturally, in some limit the notion of a classical stochastic process must be contained in the process tensor formalism. To find it, we need two assumptions. First, we need to fix the interventions $\{\mathcal{C}_k(r_k)\}$ that we apply at some time t_k, say, projective measurements of some observable R such that $\mathcal{C}(r_k)\rho_S = |r_k\rangle\langle r_k|\rho_S|r_k\rangle\langle r_k|$. Second, these interventions must be non-disturbing on average. Moreover, if the process is Markovian according to the generalized condition (31), we find Eq. (8) as it should be:

$$\begin{aligned} P(r_n|\mathbf{r}_{n-1}) &= \frac{P(\mathbf{r}_n)}{P(\mathbf{r}_{n-1})} = \frac{\mathrm{tr}_S\{\mathcal{C}(r_n)\mathcal{E}(t_n,t_{n-1})\cdots\mathcal{C}(r_1)\mathcal{E}(t_1,t_0)\mathcal{C}(r_0)\rho_S(t_0)\}}{\mathrm{tr}_S\{\mathcal{C}(r_{n-1})\mathcal{E}(t_{n-1},t_{n-2})\cdots\mathcal{C}(r_1)\mathcal{E}(t_1,t_0)\mathcal{C}(r_0)\rho_S(t_0)\}} \\ &= \frac{\langle r_n|\big[\mathcal{E}(t_n,t_{n-1})|r_{n-1}\rangle\langle r_{n-1}|\big]|r_n\rangle\ldots\langle r_1|\big[\mathcal{E}(t_1,t_0)|r_0\rangle\langle r_0|\big]|r_1\rangle\langle r_0|\rho_S(t_0)|r_0\rangle}{\langle r_{n-1}|[\mathcal{E}(t_{n-1},t_{n-2})|r_{n-2}\rangle\langle r_{n-2}|]|r_{n-1}\rangle\ldots\langle r_1|\big[\mathcal{E}(t_1,t_0)|r_0\rangle\langle r_0|\big]|r_1\rangle\langle r_0|\rho_S(t_0)|r_0\rangle} \\ &= \langle r_n|\big[\mathcal{E}(t_n,t_{n-1})|r_{n-1}\rangle\langle r_{n-1}|\big]|r_n\rangle = P(r_n|r_{n-1}). \end{aligned} \tag{34}$$

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